

Title	American Options with Uncertainty of the Stock Prices : The Discrete-Time Model (Mathematical Decision Making under Uncertainty)
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Citation	数理解析研究所講究録 (2002), 1252: 174-180
Issue Date	2002-02
URL	http://hdl.handle.net/2433/41835
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

American Options with Uncertainty of the Stock Prices: The Discrete-Time Model

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1. Introduction

A discrete-time mathematical model for American put option with uncertainty is presented, and the randomness and fuzziness are evaluated by both probabilistic expectation and λ -weighted possibilistic mean values.

2. Fuzzy stochastic processes

First we give some mathematical notations regarding fuzzy numbers. Let (Ω, \mathcal{M}, P) be a probability space, where \mathcal{M} is a σ -field and P is a non-atomic probability measure. \mathbb{R} denotes the set of all real numbers, and let $\mathcal{C}(\mathbb{R})$ be the set of all non-empty bounded closed intervals. A ‘fuzzy number’ is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [12] regarding fuzzy set theory. \mathcal{R} denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with its corresponding membership functions. The α -cut of a fuzzy number $\tilde{a} (\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \quad (\alpha \in (0, 1]) \quad \text{and} \quad \tilde{a}_0 := \text{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0, 1].$$

Hence we introduce a partial order \succeq , so called the ‘fuzzy max order’, on fuzzy numbers \mathcal{R} : Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0, 1].$$

Then (\mathcal{R}, \succeq) becomes a lattice. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order \succeq by the fuzzy number whose α -cuts are

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0, 1]. \quad (2.1)$$

An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\lambda \geq 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of \tilde{a} and \tilde{b} and the scalar multiplication $\lambda \tilde{a}$ of λ and \tilde{a} are fuzzy numbers given by

$$(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad (\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]$$

$$\text{and } (\lambda \tilde{a})_\alpha := [\lambda \tilde{a}_\alpha^-, \lambda \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0, 1].$$

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a ‘fuzzy random variable’ if the maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0, 1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ (see [10]). Next we need to introduce expectations of fuzzy random variables in order to describe an optimal stopping model in the next section. A fuzzy random variable \tilde{X} is called integrably bounded if both $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0, 1]$. Let \tilde{X} be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable \tilde{X} is defined by a fuzzy number (see [7])

$$E(\tilde{X})(x) := \sup_{\alpha \in [0, 1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}, \quad (2.2)$$

where closed intervals $E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega) dP(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega) dP(\omega) \right]$ ($\alpha \in [0, 1]$).

In the rest of this section, we introduce stopping times for fuzzy stochastic processes. Let T ($T > 0$) be an ‘expiration date’ and let $\mathbb{T} := \{0, 1, 2, \dots, T\}$ be the time space. Let a ‘fuzzy stochastic process’ $\{\tilde{X}_t\}_{t=0}^T$ be a sequence of integrably bounded fuzzy random variables such that $E(\max_{t \in \mathbb{T}} \tilde{X}_{t,0}^+) < \infty$, where $\tilde{X}_{t,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$. For $t \in \mathbb{T}$, \mathcal{M}_t denotes the smallest σ -field on Ω generated by all random variables $\tilde{X}_{s,\alpha}^-$ and $\tilde{X}_{s,\alpha}^+$ ($s = 0, 1, 2, \dots, t; \alpha \in [0, 1]$). We call $(\tilde{X}_t, \mathcal{M}_t)_{t=0}^\infty$ a fuzzy stochastic process. A map $\tau : \Omega \mapsto \mathbb{T}$ is called a ‘stopping time’ if

$$\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{M}_t \quad \text{for all } t = 0, 1, 2, \dots, T.$$

Then, the following lemma is trivial from the definitions ([11]).

Lemma 2.1. *Let τ be a stopping time. We define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \dots, T \text{ and } \omega \in \Omega.$$

Then, \tilde{X}_τ is a fuzzy random variable.

3. American put option with uncertainty of stock prices

In this section, we formulate American put option with uncertainty of stock prices by fuzzy random variables. Let $\mathbb{T} := \{0, 1, 2, \dots, T\}$ be the time space with an expiration date T ($T > 0$) similarly to the previous section, and take a probability space $\Omega := \mathbb{R}^{T+1}$. Let r ($r > 0$) be an interest rate of a bond price, which is riskless asset, and put a discount

rate $\beta = 1/(1+r)$. Define a 'stock price process' $\{S_t\}_{t=0}^T$ as follows: An initial stock price S_0 is a positive constant and stock prices are given by

$$S_t := S_0 \prod_{s=1}^t (1 + Y_s) \quad \text{for } t = 1, 2, \dots, T, \quad (3.1)$$

where $\{Y_t\}_{t=1}^T$ is a uniform integrable sequence of independent, identically distributed real random variables on $[r-1, r+1]$ such that $E(Y_t) = r$ for all $t = 1, 2, \dots, T$. The σ -fields $\{\mathcal{M}_t\}_{t=0}^T$ are defined as follows: \mathcal{M}_0 is the completion of $\{\emptyset, \Omega\}$ and \mathcal{M}_t ($t = 1, 2, \dots, T$) denote the complete σ -fields generated by $\{Y_1, Y_2, \dots, Y_t\}$.

We consider a finance model where the stock price process $\{S_t\}_{t=0}^T$ takes fuzzy values. Now we give fuzzy values by triangular fuzzy numbers for simplicity. Let $\{a_t\}_{t=0}^T$ be an \mathcal{M}_t -adapted stochastic process such that $0 < a_t(\omega) \leq S_t(\omega)$ for $\omega \in \Omega$. A 'stock price process with fuzzy values' are represented by a sequence of fuzzy random variables $\{\tilde{S}_t\}_{t=0}^T$:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega)) \quad (3.2)$$

for $t \in \mathbb{T}$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ ($x \in \mathbb{R}$) is the triangle shape function. Hence, $a_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Then, $a_t(\omega)$ should be an increasing function of the stock price $S_t(\omega)$ (see Assumption S in the next section).

Let K ($K > 0$) be a 'strike price'. The 'price process' $\{\tilde{P}_t\}_{t=0}^T$ of American put option under uncertainty is represented by

$$\tilde{P}_t(\omega) := \beta^t(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t = 0, 1, 2, \dots, T, \quad (3.3)$$

where \vee is given by (2.1), and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp number K and zero respectively. An 'exercise time' in American put option is given by a stopping time τ with values in \mathbb{T} . For an exercise time τ , we define

$$\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \dots, T, \text{ and } \omega \in \Omega. \quad (3.4)$$

Then, from Lemma 2.1, \tilde{P}_τ is a fuzzy random variable. The expectation of the fuzzy random variable \tilde{P}_τ is a fuzzy number (see (2.2))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R}, \quad (3.5)$$

where $E(\tilde{P}_\tau)_\alpha = \left[\int_\Omega \tilde{P}_{\tau,\alpha}^-(\omega) dP(\omega), \int_\Omega \tilde{P}_{\tau,\alpha}^+(\omega) dP(\omega) \right]$. In American put option, we must maximize the expected values (3.5) of the price process by stopping times τ , and we need to evaluate the fuzzy numbers (3.5) since the fuzzy max order (2.1) on \mathcal{R} is a partial order and not a linear order. In this paper, we consider the following estimation regarding the price process $\{\tilde{P}_t\}_{t=0}^T$ of American put option. Let $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda)y, \quad [x, y] \in \mathcal{C}(\mathbb{R}), \quad (3.6)$$

where λ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the evaluation of fuzzy numbers, and λ is called a ‘pessimistic-optimistic index’ and means the pessimistic degree in decision making. We call g a ‘ λ -weighting function’ and we evaluate fuzzy numbers \tilde{a} by “ λ -weighted possibilistic mean value’

$$\int_0^1 2\alpha g(\tilde{a}_\alpha) d\alpha, \quad (3.7)$$

where \tilde{a}_α is the α -cut of fuzzy numbers \tilde{a} . (see Carlsson and Fullér [1], Goetschel and Voxman [4]) When we apply a λ -weighting function g to (3.5), its evaluation follows

$$\int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha) d\alpha. \quad (3.8)$$

Now we analyze (3.8) by α -cuts technique of fuzzy numbers. The α -cuts of fuzzy random variables (3.2) are

$$\tilde{S}_{t,\alpha}(\omega) = [S_t(\omega) - (1 - \alpha)a_t(\omega), S_t(\omega) + (1 - \alpha)a_t(\omega)], \quad \omega \in \Omega, \quad (3.9)$$

and so

$$\tilde{S}_{t,\alpha}^\pm(\omega) = S_t(\omega) \pm (1 - \alpha)a_t(\omega), \quad \omega \in \Omega \quad (3.10)$$

for $t \in \mathbb{T}$ and $\alpha \in [0, 1]$. Therefore, the α -cuts of (3.3) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}_{t,\alpha}^-(\omega), \tilde{P}_{t,\alpha}^+(\omega)] := [\beta^t \max\{K - \tilde{S}_{t,\alpha}^+(\omega), 0\}, \beta^t \max\{K - \tilde{S}_{t,\alpha}^-(\omega), 0\}], \quad (3.11)$$

and we obtain $E(\max_{t \in \mathbb{T}} \sup_{\alpha \in [0,1]} \tilde{P}_{t,\alpha}^+) \leq K < \infty$ since $\tilde{S}_{t,\alpha}^-(\omega) \geq 0$, where $E(\cdot)$ is the expectation with respect to some risk-neutral equivalent martingale measure ([2],[6]). For a stopping time τ , the expectation of the fuzzy random variable \tilde{P}_τ is a fuzzy number whose α -cut is a closed interval

$$E(\tilde{P}_\tau)_\alpha = E(\tilde{P}_{\tau,\alpha}) = [E(\tilde{P}_{\tau,\alpha}^-), E(\tilde{P}_{\tau,\alpha}^+)] \quad \text{for } \alpha \in [0, 1], \quad (3.12)$$

where $\tilde{P}_{\tau(\omega),\alpha}(\omega) = [\tilde{P}_{\tau(\omega),\alpha}^-(\omega), \tilde{P}_{\tau(\omega),\alpha}^+(\omega)]$ is the α -cut of fuzzy number $\tilde{P}_\tau(\omega)$. Using the λ -weighting function g , from (3.7) the evaluation of the fuzzy random variable \tilde{P}_τ is given by the integral

$$\int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha. \quad (3.13)$$

Put the value by $\mathbf{P}(\tau)$. Then, from (2.2), the terms (3.8) and (3.13) coincide:

$$\mathbf{P}(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha = \int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha) d\alpha. \quad (3.14)$$

Therefore $\mathbf{P}(\tau)$ means an evaluation of the expected price of American put option when τ is an exercise time. Further, we have the following equality.

Lemma 3.1. For a stopping time τ ($\tau \leq T$), it holds that

$$P(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha = \int_0^1 2\alpha E(g(\tilde{P}_{\tau,\alpha})) d\alpha = E\left(\int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\cdot)) d\alpha\right). \quad (3.15)$$

We put the ‘optimal expected price’ by

$$V := \sup_{\tau: \tau \leq T} P(\tau) = \sup_{\tau: \tau \leq T} \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) d\alpha. \quad (3.16)$$

In the next section, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find a stopping time τ^* ($\tau^* \leq T$) and the optimal expected price V such that

$$P(\tau^*) = V, \quad (3.17)$$

where V is given by (3.16).

Then, τ^* is called an ‘optimal exercise time’.

4. The optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price V and the optimal exercise time τ^* , by using dynamic programming approach. Now we introduce an assumption.

Assumption S. The stochastic process $\{a_t\}_{t=0}^T$ is represented by

$$a_t(\omega) := cS_t(\omega), \quad t = 0, 1, 2, \dots, T, \quad \omega \in \Omega,$$

where c is a constant satisfying $0 < c < 1$.

Assumption S is reasonable since $a_t(\omega)$ means a size of fuzziness and it should depend on the volatility and the stock price $S_t(\omega)$ because one of the most difficulties is estimation of the actual volatility ([8, Sect.7.5.1]). In this model, we represent by c the fuzziness of the volatility, and we call c a ‘fuzzy factor’ of the process. From now on, we suppose that Assumption S holds. For a stopping time τ ($\tau \leq T$), we define a random variable

$$\Pi_\tau(\omega) := \int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\omega)) d\alpha, \quad \omega \in \Omega. \quad (4.1)$$

From Lemma 3.1, $P(\tau) = E(\Pi_\tau)$ is the evaluated price of American put option when τ is an exercise time. Then we have the following representation about (4.1).

Lemma 4.1. For a stopping time τ ($\tau \leq T$), it holds that

$$\Pi_\tau(\omega) = \beta^{\tau(\omega)} f^P(S_\tau(\omega)), \quad \omega \in \Omega, \quad (4.2)$$

where f^P is a function on $(0, \infty)$ such that

$$f^P(y) := \begin{cases} K - y - \frac{1}{3}cy(2\lambda - 1) + \lambda\varphi^1(y) & \text{if } 0 < y < K \\ (1 - \lambda)\varphi^2(y) & \text{if } y \geq K, \end{cases} \quad (4.3)$$

and

$$\varphi^1(y) := \frac{1}{(cy)^2}((-K + y + cy) \max\{0, -K + y + cy\}^2 - \frac{2}{3} \max\{0, -K + y + cy\}^3), \quad y > 0, \quad (4.4)$$

$$\varphi^2(y) := \frac{1}{(cy)^2}((K - y + cy) \max\{0, K - y + cy\}^2 - \frac{2}{3} \max\{0, K - y + cy\}^3), \quad y > 0. \quad (4.5)$$

Now we give an optimal stopping time for Problem P and we discuss an iterative method to obtain the optimal expected price V in (3.16). To analyze the optimal fuzzy price V , we put

$$V_t^P(y) = \sup_{\tau: t \leq \tau \leq T} E(\beta^{-t} \Pi_\tau | S_t = y) \quad (4.6)$$

for $t = 0, 1, 2, \dots, T$ and an initial stock price y ($y > 0$). Then we note that $V = V_0^P(y)$.

Theorem 4.1 (Optimality equation).

- (i) The optimal expected price $V = V_0^P(y)$ with an initial stock price y ($y > 0$) is given by the following backward recursive equations (4.7) and (4.8):

$$V_t^P(y) = \max\{\beta E(V_{t+1}^P(y(1 + Y_1))), f^P(y)\}, \quad t = 0, 1, \dots, T - 1, \quad y > 0, \quad (4.7)$$

$$V_T^P(y) = f^P(y), \quad y > 0. \quad (4.8)$$

- (ii) Define a stopping time

$$\tau^P(\omega) := \inf\{t \in \mathbb{T} \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}, \quad \omega \in \Omega, \quad (4.9)$$

where the infimum of the empty set is understood to be T . Then, τ^P is an optimal exercise time for Problem P, and the optimal value of American put option is

$$V = V_0^P(y) = P(\tau^P) \quad (4.10)$$

for an initial stock price $y > 0$.

5. A numerical example

Now we give a numerical example to illustrate our idea in Sections 3 and 4.

Example 5.1. We consider CRR type American put option model (see Ross [8, Sect.7.4]). Put an expiration date $T = 10$, an interest rate of a bond $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 30$ and a strike price $K = 35$. Assume that

$\{Y_t\}_{t=1}^T$ is a uniform sequence of independent, identically distributed real random variables such that

$$Y_t := \begin{cases} e^\sigma - 1 & \text{with probability } p \\ e^{-\sigma} - 1 & \text{with probability } (1 - p) \end{cases}$$

for all $t = 1, 2, \dots, T$, where $\sigma = 0.25$ and $p = (1 + r - e^{-\sigma})/(e^\sigma - e^{-\sigma})$. Then we have $E(Y_t) = r$. The corresponding optimal exercise time is given by

$$\tau^P(\omega) = \inf\{t \in \mathbb{T} \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}.$$

In the following Table, the optimal expected price $V = V_0^P(y)$ at initial stock price $y = 30$ changes with the pessimistic-optimistic index λ of the λ -weighting function g .

Table. The optimal expected price $V = V_0^P(y)$ at initial stock prices $y = 30$.

λ	1/3	1/2	2/3
V	7.48169	7.39649	7.31130

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